Pergamon

J. Appl. Maths Mechs, Vol. 58, No. 2, pp. 355–362, 1994

0021–8928/94 $24.00+0.00

0021-8928(94)E0059-J

GREEN'S FUNCTION FOR A COMPOSITE PIEZOCERAMIC PLANE WITH A CRACK BETWEEN PHASES†

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(*Received* 11 *May* 1993)

Green's function for a composite (piecewise-uniform) piezoceramic plane with a crack between the phases is constructed explicitly. It is assumed that the crack edges are free from mechanical loads and the normal component of the electric induction vector and the tangential component of the electric field strength vector are continuous along the crack. The known representation of the solution of the problem of electroelasticity using six functions analytic in half-planes is used, Green's function for a composite plane without a crack being constructed in the first place. The solution of the fundamental problem is found using analytic continuation and is reduced to matrix Riemann problem in a finite interval. The stress intensity factors at the crack tips are also determined.

Green's function was constructed in [1] for a homogeneous piezoceramic plane with a rectilinear cut.

1. Consider an unbounded medium consisting of two different piezoceramic half-planes attached to one another along the sections $x_3=0$, $|x_1|\geqslant a$ (Fig. 1). There is a gap along the section $x_3=0$, $|x_1|<a$, which will be treated as a crack between the phases. Suppose that either a concentrated force $P=(P_1, P_3)$ is applied at a point (x_{10}, x_{30}) in the upper half-plane or a concentrated electric charge ρ is placed at that point. Denoting the quantities corresponding to the half-plane r by a superscript r $(r=1, 2)$ we write the mechanical and electrical boundary conditions on the x_1-axis in the form

$$\sigma_{i3}^{(1)}=\sigma_{i3}^{(2)},\quad du_i^{(1)}=du_i^{(2)},\quad |x_1|\geqslant a\quad (i=1,3)$$
$$E_1^{(1)}=E_1^{(2)},\quad D_3^{(1)}=D_3^{(2)},\quad -\infty<x_1<\infty \tag{1.1}$$

$$\sigma_{i3}^{(1)+}=0,\quad \sigma_{i3}^{(2)-}=0,\quad |x_1|<a \tag{1.2}$$

where σ_{i3}, u_i, E_1, D_3 are the components of the stress tensor, the displacement vector, and the electric field strength and induction vectors, respectively. By $\sigma^{\pm}$ we mean the limiting values of σ on the upper edge (the plus sign) and lower edge (the minus sign) of the crack.

We assume that the initial piezoceramic polarization vector on the upper and lower half-planes is parallel to the x_3-axis. In this case the components of the vector $U=\{U_k\}=\{\sigma_{33}, -\sigma_{13}, u_1', u_3', E_1, -D_3\}$ have the form [2]

$$U_k=2\operatorname{Re}\sum_{\nu=1}^{3} c_{k\nu}\Phi_\nu(z_\nu)\quad (k=1,2,..,6) \tag{1.3}$$

†*Prikl. Mat. Mekh.* Vol. 58, No. 2, pp. 159–166, 1994.

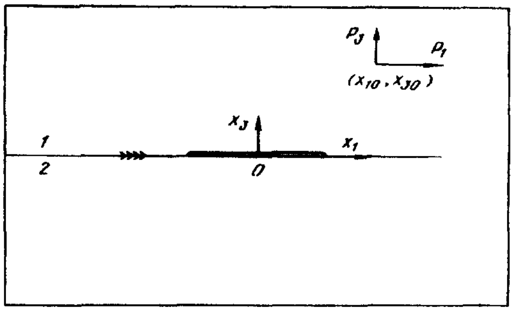

Fig. 1.

$$c_{1\nu} = \gamma_\nu,\; c_{2\nu} = \gamma_\nu\mu_\nu,\; c_{3\nu} = p_\nu,\; c_{4\nu} = q_\nu,\; c_{5\nu} = \lambda_\nu,\; c_{6\nu} = r_\nu$$

$$\lambda_\nu = \varepsilon_{11} + \varepsilon_{33}\mu_\nu^2,\; p_\nu = s_{11}\gamma_\nu\mu_\nu^2 + s_{13}\gamma_\nu + d_{31}\lambda_\nu\mu_\nu$$

$$q_\nu = s_{33}\gamma_\nu / \mu_\nu + s_{13}\gamma_\nu\mu_\nu + d_{33}\lambda_\nu$$

$$r_\nu = \varepsilon_{11}\lambda_\nu / \mu_\nu - d_{15}\gamma_\nu,\; z_\nu = x_1 + \mu_\nu x_3$$

$$\lambda_\nu = (d_{15} - d_{33})\mu_\nu - d_{31}\mu_\nu^3$$

Here μ_ν ($\mathrm{Im}\mu_\nu > 0$, $\nu = 1, 2, 3$) are the roots of the characteristic equation

$$\{s_{33} + (s_{44} + 2s_{13})\mu^2 + s_{11}\mu^4\}(\varepsilon_{11} + \varepsilon_{33}\mu^2) - \mu^2(d_{15} - d_{33} - d_{31}\mu^2)^2 = 0$$

The coefficients $s_{ij} = s_{ij}^E$ are the elastic compliances for a constant value of the electric field, d_{ij} are the piezoelectric constants, and $\varepsilon_{ii} = \varepsilon_{ii}^T$ are the permittivities for a constant value of the mechanical stress [3, 4].

Hence we arrive at the boundary-value problem for determining the six functions $\Phi_\nu^{(r)}(z_\nu^{(r)})$, each of which is analytic in its plane $z_\nu^{(r)} = x_1 + \mu_\nu^{(r)}x_3$ ($\nu = 1, 2, 3$; $r = 1, 2$), from the conditions ensuring that the components U_k ($k = 1, 2, \ldots, 6$) can be extended by continuity across the intervals $|x_1| \geqslant a$ and the conditions $U_1^\pm = 0$, $U_2^\pm = 0$, $U_5^{(1)} = U_5^{(2)}$, $U_6^{(1)} = U_6^{(2)}$ on the edges of the interval $[-a, a]$. This problem should be solved by means of the analytic continuation of the appropriate functions followed by reducing it to Riemann problems. However, to begin with, one must construct the fundamental solution for a composite plane without a gap.

2. Consider a piecewise uniform plate consisting of two different piezoceramic half-planes attached to one another along the entire length of the x_1-axis. Suppose that either a concentrated force $P = (P_1, P_3)$ acts at a point (x_{10}, x_{30}) in the upper plane or a charge ρ is placed at that point. By generalizing the reflection method [5] and using the results of [6], we can represent the solution in the form

$$\Phi_\nu^{(1)}(z_\nu^{(1)}) = \frac{A_\nu^{(1)}}{z_\nu^{(1)} - z_{\nu 0}^{(1)}} - \sum_{m=1}^{3} \frac{\overline{\alpha_{\nu m}^{(1)}}\,\overline{A_m^{(1)}}}{z_\nu^{(1)} - \bar{z}_{m0}^{(1)}} \quad (\nu = 1,2,3) \tag{2.1}$$

$$\Phi_\nu^{(2)}(z_\nu^{(2)}) = \sum_{m=1}^{3} \frac{\alpha_{\nu+3,m}^{(1)} A_m^{(1)}}{z_\nu^{(2)} - z_{m0}^{(1)}};\quad z_\nu^{(1)} = x_1 + \mu_\nu^{(1)}x_3,\quad z_{m0}^{(1)} = x_{10} + \mu_m^{(1)}x_{30}$$

Here

$$2\,\mathrm{Im} \sum_{m=1}^{3} A_m^{(1)}(\mu_m^{(1)})^{n-1} = -\frac{B_n}{h} \quad (n = 0,1,\ldots,5) \tag{2.2}$$

$$B_0 = \frac{P_1}{2\pi\varepsilon_{11}^{(1)}s_{33}^{(1)}}\left(s_{13}^{(1)} - \frac{1}{\Delta_1}\varepsilon_{33}^{(1)}s_{33}^{(1)}d_{31}^{(1)}\right)$$

$$B_1 = -\frac{P_3}{2\pi\varepsilon_{11}^{(1)}\Delta_1}(\varepsilon_{33}^{(1)}d_{15}^{(1)} + \varepsilon_{11}^{(1)}d_{31}^{(1)}) - \frac{\rho\varepsilon_{33}^{(1)}}{2\pi\varepsilon_{11}^{(1)}\Delta_1}$$

$$B_2 = \frac{P_1 d_{31}^{(1)}}{2\pi\Delta_1},\quad B_3 = \frac{P_3 d_{33}^{(1)} + \rho}{2\pi\Delta_1},\quad B_4 = \frac{P_1}{2\pi\Delta_1}(d_{15}^{(1)} - d_{33}^{(1)})$$

$$B_5 = \frac{P_3}{2\pi\Delta_1\Delta_2}(d_{33}^{(1)}\Delta_3 - s_{13}^{(1)}\Delta_1) + \frac{\rho\Delta_3}{2\pi\Delta_1\Delta_2}$$

$$\Delta_1 = (d_{33}^{(1)} - d_{15}^{(1)})\varepsilon_{33}^{(1)} - d_{31}^{(1)}\varepsilon_{11}^{(1)},\quad \Delta_2 = s_{11}^{(1)}\varepsilon_{33}^{(1)} - (d_{31}^{(1)})^2$$

$$\Delta_3 = (d_{33}^{(1)} - d_{15}^{(1)})d_{31}^{(1)} - s_{11}^{(1)}\varepsilon_{11}^{(1)}$$

$\overline{A}$ is the complex conjugate of A and the constants $A_m^{(1)}$ can be determined from the system of equations (h is the thickness of the plate).

The determinant of this system is

$$\Delta = -8i\,\mathrm{Im}\,\mu_1\,\mathrm{Im}\,\mu_2\,\mathrm{Im}\,\mu_3|\mu_1\mu_2\mu_3|^{-2}\times$$
$$\times\ |(\mu_2-\mu_1)(\mu_3-\mu_1)(\mu_3-\mu_2)(\mu_2-\overline{\mu}_1)(\mu_3-\overline{\mu}_1)(\mu_3-\overline{\mu}_2)|\neq 0$$

To compute $\alpha_{\nu m}^{(1)}$ we invoke the boundary conditions (1.1) on the whole x_1-axis. Substitution of (2.1) into these conditions leads to the following relationships uniquely defining the matrix $\|\alpha_{\nu m}^{(1)}\|$ ($\nu = 1, 2, \ldots, 6$; $m = 1, 2, 3$)

$$\sum_{\nu=1}^{3}(\overline{c_{k\nu}^{(1)}\alpha_{\nu m}^{(1)}} + c_{k\nu}^{(2)}\alpha_{\nu+3,m}^{(1)}) = c_{km}^{(1)}\quad (m = 1,2,3;\ k = 1,2,\ldots,6). \tag{2.3}$$

This completes the construction of the fundamental solution of (2.1).

If the concentrated force is applied at a point $(x_{10},\ x_{30})$ on the lower half-plane, the fundamental solution has the form

$$\Phi_\nu^{(1)}(z_\nu^{(1)}) = \sum_{m=1}^{3}\frac{\alpha_{\nu+3,m}^{(2)}A_m^{(2)}}{z_\nu^{(1)} - z_{m1}^{(2)}},\quad z_{m1}^{(2)} = x_{10}' + \mu_m^{(2)}x_{30}' \tag{2.4}$$

$$\Phi_\nu^{(2)}(z_\nu^{(2)}) = \frac{A_\nu^{(2)}}{z_\nu^{(2)} - z_{\nu1}^{(2)}} - \sum_{m=1}^{3}\frac{\overline{\alpha_{\nu m}^{(2)}}\ \overline{A_m^{(2)}}}{z_\nu^{(2)} - \overline{z}_{m1}^{(2)}}\quad (\nu = 1,2,3)$$

where $A_\nu^{(2)}$ and $\alpha_{\nu m}^{(2)}$ can, respectively, be determined from Eqs (2.2) and (2.3) with the superscripts 1 and 2 interchanged.

3. We will seek a solution of the fundamental problem posed in Section 1 in the form

$$W_\nu^{(r)}(z_\nu^{(r)}) = \Phi_\nu^{(r)}(z_\nu^{(r)}) + \Psi_\nu^{(r)}(r_\nu^{(r)})\quad (\nu = 1,2,3;\ r = 1,2) \tag{3.1}$$

the functions $\Phi_\nu^{(r)}$ $(z_\nu^{(r)})$ being given by (2.1)–(2.3), while $\Psi_\nu^{(r)}$ $(z_\nu^{(r)})$, which account for the perturbations due to the gap between the phases, are to be determined.

We shall use the idea of analytic continuation of $\Psi_\nu^{(r)}$ $(z_\nu^{(r)})$ as follows. We continue $\Psi_\nu^{(1)}(z)$ $(\Psi_\nu^{(2)}(z))$ analytically across the intervals $|x_1|\geq a$ into the lower (upper) half-plane with the aid of the functions $\overline{\Psi_\nu^{(1)}}(z)$ $(\overline{\Psi_\nu^{(2)}}(z))$, which satisfy the relationship $\overline{\Psi_\nu^{(1)+}(x_1)} = \overline{\Psi_\nu^{(1)-}}(x_1)$

$(\overline{\Psi_\nu^{(2)-}(x_1)} = \overline{\Psi}_\nu^{(2)+}(x_1))$. We use the definition $\overline{\Psi}(z)=\overline{\Psi(\overline{z})}$ and we denote by $\Psi^\pm(x_1)$ the corresponding limiting values of $\Psi(z)$. Taking all this into account, we find from (1.3) that

$$U_k^+(x_1)=\sum_{\nu=1}^{3}\left\{c_{k\nu}^{(1)}\Psi_\nu^{(1)+}(x_1)+\overline{c_{k\nu}^{(1)}}\,\overline{\Psi_\nu^{(1)-}}(x_1)\right\}+2\,\mathrm{Re}\sum_{\nu=1}^{3}c_{k\nu}^{(1)}\Phi_\nu^{(1)}(x_1) \tag{3.2}$$

$$U_k^-(x_1)=\sum_{\nu=1}^{3}\left\{c_{k\nu}^{(2)}\Psi_\nu^{(2)-}(x_1)+\overline{c_{k\nu}^{(2)}}\,\overline{\Psi_\nu^{(2)+}}(x_1)\right\}+2\,\mathrm{Re}\sum_{\nu=1}^{3}c_{k\nu}^{(2)}\Phi_\nu^{(2)}(x_1)$$

We introduce the functions

$$F_k(z)=\begin{cases} f_k(z)=\sum_{\nu=1}^{3}\{c_{k\nu}^{(1)}\Psi_\nu^{(1)}(z)-\overline{c_{k\nu}^{(2)}}\,\overline{\Psi_\nu^{(2)}}(z)\}, & x_3>0\\ f_k(z)=\sum_{\nu=1}^{3}\{c_{k\nu}^{(2)}\Psi_\nu^{(2)}(z)-\overline{c_{k\nu}^{(1)}}\,\overline{\Psi_\nu^{(1)}}(z)\}, & x_3<0\ (k=1,2,\dots,6)\end{cases} \tag{3.3}$$

By (1.3) and the fact that U_k $(k=1,\,2,5,\,6)$ can be extended by continuity across the cut, we get

$$f_k^+(x_1)-f_k^-(x_1)=U_k^+(x_1)-U_k^-(x_1)=0 \tag{3.4}$$

It follows that the functions $f_k(z)$ $(k=1,\,2,\,5,\,6)$ with elements $f_k(z)$ for $\mathrm{Im}\,z>0$ and $f_k(z)$ for $\mathrm{Im}\,z<0$ are analytic in the whole z-plane and can be set equal to zero.

Relationships (3.3) imply the limiting equalities

$$\begin{aligned}&\sum_{\nu=1}^{3}\{c_{k\nu}^{(1)}\Psi_\nu^{(1)+}(x_1)-\overline{c_{k\nu}^{(2)}}\,\overline{\Psi_\nu^{(2)+}}(x_1)\}=f_k^+(x_1)\\ &\sum_{\nu=1}^{3}\{\overline{c_{k\nu}^{(1)}}\,\overline{\Psi_\nu^{(1)-}}(x_1)-c_{k\nu}^{(2)}\Psi_\nu^{(2)-}(x_1)\}=-f_k^-(x_1)\ (k=1,2,\dots,6)\end{aligned} \tag{3.5}$$

which, by (3.2), (3.4), and the boundary conditions $U_k^+=U_k^-=0$ $(k=1,\,2)$ lead to the following matrix Riemann problem in the interval $(-a,a)$

$$Bf^+(x_1)-\overline{B}f^-(x_1)=R(x_1) \tag{3.6}$$

$$B=\|b_{ij}\|=\begin{Vmatrix} b_{11}, & b_{12}, & \dots, & b_{16}\\ b_{21}, & b_{22}, & \dots, & b_{26}\end{Vmatrix}=AC_*^{-1},\quad R(x_1)=\begin{Vmatrix}N_1\\ N_2\end{Vmatrix}$$

$$A=\begin{Vmatrix} c_{11}^{(1)}, & c_{12}^{(1)}, & c_{13}^{(1)}, & \overline{c_{11}^{(2)}}, & \overline{c_{12}^{(2)}}, & \overline{c_{13}^{(2)}}\\ c_{21}^{(1)}, & c_{22}^{(1)}, & c_{23}^{(1)}, & \overline{c_{21}^{(2)}}, & \overline{c_{22}^{(2)}}, & \overline{c_{23}^{(2)}}\end{Vmatrix},\quad N_k=-4\,\mathrm{Re}\sum_{\nu=1}^{3}c_{k\nu}^{(2)}\Phi_\nu^{(2)}(x_1)$$

$$C_*=\begin{Vmatrix} c_{11}^{(1)}, & c_{12}^{(1)}, & c_{13}^{(1)}, & -\overline{c_{11}^{(2)}}, & -\overline{c_{12}^{(2)}}, & -\overline{c_{13}^{(2)}}\\ c_{21}^{(1)}, & c_{22}^{(1)}, & c_{23}^{(1)}, & -\overline{c_{21}^{(2)}}, & -\overline{c_{22}^{(2)}}, & -\overline{c_{23}^{(2)}}\\ \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots\\ \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots\\ c_{61}^{(1)}, & c_{62}^{(1)}, & c_{63}^{(1)}, & -\overline{c_{61}^{(2)}}, & -\overline{c_{62}^{(2)}}, & -\overline{c_{63}^{(2)}}\end{Vmatrix},\quad f^\pm(x_1)=\begin{Vmatrix}0\\0\\f_3^\pm\\f_4^\pm\\0\\0\end{Vmatrix}$$

Now we choose $(\rho_{k1},\,\rho_{k2})$ and λ_k $(k=1,\,2)$ to be the eigenvectors and eigenvalues, respectively, of the following homogeneous systems

$$\rho_{k1}(\overline{b}_{13}+\lambda_k b_{13})+(-1)^{k-1}\rho_{k2}(\overline{b}_{23}+\lambda_k b_{23})=0\ (k=1,2) \tag{3.7}$$

$$\rho_{k1}(\overline{b}_{14}+\lambda_k b_{14})+(-1)^{k-1}\rho_{k2}(\overline{b}_{24}+\lambda_k b_{24})=0\ (k=1,2) \tag{3.8}$$

Then (3.6) can be reduced to two scalar Riemann problems

$$\rho_k^+(x_1)+\lambda_k\rho_k^-(x_1)=N_k^*(x_1),\ |x_1|<a\ (k=1,2) \tag{3.9}$$
$$\rho_k(x_1)=\sum_{j=3}^{4} d_{kj}f_j(x_1),\ d_{kj}=\frac{2i\,\mathrm{Im}(b_{1j}\bar b_{2j})}{\bar b_{2j}+\lambda_k b_{2j}}\ (j=3,4)$$
$$N_k^*=\rho_{k1}N_1(x_1)+(-1)^{k-1}\rho_{k2}N_2(x_1)$$

The eigenvalues λ_1 and λ_2 appearing in (3.9) can be found from the conditions for the non-trivial solvability of system (3.7), (3.8). these conditions can be reduced to the quadratic equation

$$\lambda_{1,2}^2+2r_1\lambda_{1,2}+r_2=0 \tag{3.10}$$
$$r_1=\frac{\mathrm{Re}(\bar b_{13}b_{24}-\bar b_{23}b_{14})}{b_{13}b_{24}-b_{14}b_{23}},\ r_2=\frac{\bar b_{13}\bar b_{24}-\bar b_{23}\bar b_{14}}{b_{13}b_{24}-b_{14}b_{23}}$$

The roots of Eq. (3.10) can of course be represented as

$$\lambda_1=\frac{1}{R}e^{-i\Theta},\ \lambda_2=\mathrm{Re}^{-i\Theta},\ \Theta=-\frac{1}{2}\arg r_2 \tag{3.11}$$
$$\mathrm{Im}\,R=0,\ 0\leqslant\Theta<2\pi$$

Solving (3.9), we find that [7]

$$\rho_k(z)=\{D_k(z)+M_k\}X_k(z)\ (k=1,2) \tag{3.12}$$
$$D_k(z)=\frac{1}{2\pi i}\int_{-a}^{a}\frac{N_k^*(x)dx}{(x-z)X_k(x)},\ \gamma_2=\bar\gamma_1$$
$$X_k(z)=(z+a)^{-\gamma_k}(z-a)^{\gamma_k-1},\ \gamma_1=\frac{1}{2}-\frac{\Theta}{2\pi}+\frac{i}{2\pi}\ln R$$

where $X_k(x)=X_k^+(x)$ are the values of the canonical functions $X_k(z)$ on the upper edge of the cut, M_k being arbitrary complex constants.

To fix these constants one must ensure, first, that the displacement vector can be extended by continuity across the intervals $|x_1|\geqslant a$ (the boundary conditions involve the derivatives with respect to displacements). This condition will be satisfied if M_1 and M_2 are connected by $\overline{M}_2=\lambda_1 M_1$. Next, we state the uniqueness conditions for the displacements in the composite plane

$$\int_c du_i=-\int_{-a}^{a}d[u_i]=0\ (i=1,2) \tag{3.13}$$

where c is an arbitrary closed contour surrounding the interval $[-a,a]$ on the x_1-axis and $[u]$ is the jump of u when passing across the cut.

Introducing here the expression for $u_i'(x_1)$ from (1.3) and taking (3.1) and (3.12) into account, we find after some reduction that

$$M_i=\frac{2}{1+\lambda_i}\sum_{m=1}^{3}(A_m^{(1)}\Omega_{i1}^{(m)}+\overline{A_m^{(1)}\Omega_{i2}^{(m)}})\ (i=1,2) \tag{3.14}$$

$$\Omega_{11}^{(m)}=\Omega_{22}^{(m)}=-\sum_{\nu=1}^{3}\sum_{j=1}^{2}\rho_{1j}c_{j\nu}^{(2)}\alpha_{\nu+3,m}^{(1)}\ (m=1,2,3)$$

$$\Omega_{21}^{(m)} = \Omega_{12}^{(m)} = \sum_{\nu=1}^{3} \sum_{j=1}^{2} (-1)^j \rho_{2j} c_{j\nu}^{(2)} \alpha_{\nu+3,m}^{(1)} \quad \rho_{k1} = 1$$

$$\rho_{k2} = (-1)^k (\bar{b}_{13} + \lambda_k b_{13})(\bar{b}_{23} + \lambda_k b_{23})^{-1} \quad (k = 1, 2)$$

Now we express the desired functions in terms of $f_k(z)$ from (3.3). We have

$$\Psi_\nu^{(1)}(z) = \sum_{j=3}^{4} l_{\nu j} f_j(z), \ \operatorname{Im} z > 0 \ (\nu = 1, 2, 3)$$

$$\Psi_\nu^{(2)}(z) = -\sum_{j=3}^{4} \bar{l}_{\nu+3,j} f_j(z), \ \operatorname{Im} z < 0$$

where l_{mj} $(m = 1, 2, \ldots, 6;\ j = 3, 4)$ are the matrix elements of C_*^{-1}.

By substituting the functions $f_3(z)$, $f_4(z)$ found from (3.12), carrying but the necessary quadratures, and using (3.14), we finally find that

$$\Psi_\nu^{(r)}(z_\nu^{(r)}) = \sum_{m=1}^{3} \sum_{n=1}^{2} \left\{ \left(\frac{1 - X_n(z_\nu^{(r)}) X_n^{-1}(z_{m0}^{(1)})}{z_\nu^{(r)} - z_{m0}^{(1)}} + X_n(z_\nu^{(r)}) \right) \Omega_{n1}^{(m)} \times \right.$$

$$\left. \times\ \gamma_{n\nu}^{(r)} A_m^{(1)} + \left(\frac{1 - X_n(z_\nu^{(r)}) X_n^{-1}(\overline{z_{m0}^{(1)}})}{z_\nu^{(r)} - \bar{z}_{m0}^{(1)}} + X_n(z_\nu^{(r)}) \right) \overline{\Omega_{n2}^{(m)}} \gamma_{n\nu}^{(r)} \overline{A_m^{(1)}} \right\} \tag{3.15}$$

$$\gamma_{n\nu}^{(r)} = \frac{2L_{n\nu}^{(r)}}{(1 + \lambda_n) d_0}, \quad d_0 = d_{13} d_{24} - d_{14} d_{23}$$

$$L_{n\nu}^{(1)} = \sum_{j=3}^{4} (-1)^{n-j} m_{nj} l_{\nu j}, \quad L_{n\nu}^{(2)} = -\sum_{j=3}^{4} (-1)^{n-j} m_{nj} \bar{l}_{\nu+3,j}$$

$$m_{n3} = \frac{d_{14} d_{24}}{d_{n4}}, \quad m_{n4} = \frac{d_{13} d_{23}}{d_{n3}} \quad (\nu = 1, 2, 3;\ r = 1, 2)$$

It follows that Green's function for a composite piezoceramic plane with a crack between the phases can be found explicitly from (3.1), (2.1) and (3.15). The analytic representation of Green's function implies that the mechanical stresses as well as the components of the electric field at the crack tips between the phases have power-like singularities, amplified by the oscillations in small neighbourhoods of $x_1 = \pm a$. This effect is also present in isotropic composite media [8].

4. To determine the stress intensity factors at the tips of the crack between the phases we leave in (3.15) only those terms that contain the canonical functions $X_n(z)$. Then in the vicinity of $x_1 = \pm a$ we obtain

$$\{V_k^{(r)}\} = 4 \operatorname{Re} \sum_{m=1}^{3} \sum_{n=1}^{2} A_m^{(1)} \left\{ \frac{(x_1 - a)^{\gamma_n - 1}}{(2a)^{\gamma_n}} \Omega_{1n}^{(m)} D_{kn} \left[\left(\frac{z_{m0}^{(1)} + a}{z_{m0}^{(1)} - a} \right)^{\gamma_n} + 1 \right] \right\} + O(1) \tag{4.1}$$

for $x_1 > a$ and

$$\{V_k^{(r)}\} = -4 \operatorname{Re} \sum_{m=1}^{3} \sum_{n=1}^{2} A_m^{(1)} \left\{ \frac{(x_1 + a)^{-\gamma_n}}{(2a)^{1-\gamma_n}} \Omega_{1n}^{(m)} D_{kn} \left[\left(\frac{z_{m0}^{(1)} - a}{z_{m0}^{(1)} + a} \right)^{1-\gamma_n} + 1 \right] \right\} + O(1)$$

$$V_1^{(r)} = \sigma_{33}^{(r)}, \quad V_2^{(r)} = -\sigma_{13}^{(r)} \quad (r = 1, 2)$$

$$D_{kl}^{(1)} = \sum_{\nu=1}^{3} \sum_{j=3}^{4} (-1)^{j-1} c_{k\nu}^{(1)} l_{\nu j} m_{1j}, \quad D_{k2} = \overline{D}_{kl}, \quad D_{kl} = D_{kl}^{(1)} d_0^{-1}$$

for $x_1 < -a$.

As expected, the resulting expressions for $V_k^{(r)}$ are independent of r. In accordance with the above asymptotic forms, we express the mechanical stress intensity factors as follows. At the tip $x_1 = a$

$$K_j^+ = \lim\left\{\left(\frac{x_1}{a} - 1\right)^{1/2+\Theta/(2\pi)} \sigma_j(x_1)\right\} = \frac{\Lambda}{ah}\langle K_j^+\rangle, \quad x_1 \to a \ (x_1 > a) \tag{4.2}$$

$$\sigma_1(x_1) = \sigma_{33}, \quad \sigma_2(x_1) = \sigma_{13}, \quad B_m^{(1)} = h\left[P_1^2 + P_3^2 + \left(\frac{\rho}{d_{33}^{(1)}}\right)^2\right]^{-\frac{1}{2}} A_m^{(1)}$$

$$\langle K_j^+\rangle = 4\,\mathrm{Re}\sum_{m=1}^{3} B_m^{(1)} \sum_{n=1}^{2} \frac{\Omega_{1n}^{(m)} D_{jn}}{2^{\gamma_n}} \left\{\left(\frac{z_{m0}^{(1)} + a}{z_{m0}^{(1)} - a}\right)^{\gamma_n} + 1\right\} \quad (j = 1, 2)$$

At the tip $x_1 = -a$

$$K_j^- = \lim\left\{\left|\frac{x_1}{a} + 1\right|^{1/2-\Theta/(2\pi)} \sigma_j(x_1)\right\} = \frac{\Lambda}{ah}\langle K_j^-\rangle, \quad x_1 \to -a \ (x_1 < -a) \tag{4.3}$$

$$\langle K_j^-\rangle = -4\,\mathrm{Re}\sum_{m=1}^{3} B_m^{(1)} \sum_{n=1}^{2} \frac{\Omega_{1n}^{(m)} D_{jn}}{2^{1-\gamma_n}} \left\{\left(\frac{z_{m0}^{(1)} - a}{z_{m0}^{(1)} + a}\right)^{1-\gamma_n} + 1\right\}$$

Here $\Lambda = P$ for $P = \sqrt{P_1^2 + P_3^2} > 0$ and $\rho = 0$, and $\Lambda = \rho / d_{33}^{(1)}$ for $\rho \neq 0$ and $P = 0$.

When evaluating these limits it was assumed [8] that $\mathrm{Im}\gamma_1 \cdot \ln|x_1 \pm a| \approx 0$. Computations confirm this approximate equality.

Nevertheless, the solution remains physically incorrect in "microscopic" neighbourhoods of the tips because the stresses change their sign an infinite number of times.

5. The above approach can also be extended without major modifications to the case of an anisotropic composite plane with a crack on the dividing boundary between the phases in a Cartesian system x_1Ox_2. Green's function is given by (3.15) with summation with respect to m from 1 to 2, and the coefficients $c_{k\nu}$ (k, $\nu = 1, 2$) are as follows: $c_{1\nu} = 1$, $c_{2\nu} = \mu_\nu$, $c_{3\nu} = \alpha_{11}\mu_\nu^2 - a_{16}\mu_\nu + a_{12}$, $c_{4\nu} = a_{12}\mu_\nu - a_{26} + a_{12}/\mu_\nu$ (a_{jk} are the elasticity parameters of the material, and μ_ν are the corresponding eigenvalues [9]). In (4.2) and (4.3) by σ_1 and σ_2 we now mean, respectively, the components σ_{22} and σ_{12} of the stress tensor ($\rho = 0$).

6. As an example consider a PZT-5 (the upper half-plane), $BaTiO_3$ (the lower half-plane) pair with a crack $x_3 = 0$, $-a < x_1 < a$ between the phases and a concentrated force (P_1, P_3) or charge ρ at the point $x_{10} = 0$, $x_{30} = H > 0$. The dependence of $\langle K_1^\pm\rangle$ on $\varepsilon = H/a$ is shown in Fig. 2, in which curves 1–3 are constructed for the cases $P_1 \neq 0$, $P_3 = \rho = 0$; $\rho \neq 0$, $P_1 = P_3 = 0$ and $P_3 \neq 0$, $P_1 = \rho = 0$, respectively. The solid lines represent $\langle K_1^-\rangle$ and the dashed ones $\langle K_1^+\rangle$. The analogous results for $\langle K_2^\pm\rangle$ are presented in Fig. 3.

The results are altered if the components of the pair are interchanged. In this case the data for $\langle K_1^\pm\rangle$ are presented in Fig. 4 We put $P/h = 1$, $\rho/(d_{33}^{(1)}h) = 1$ in the computations.

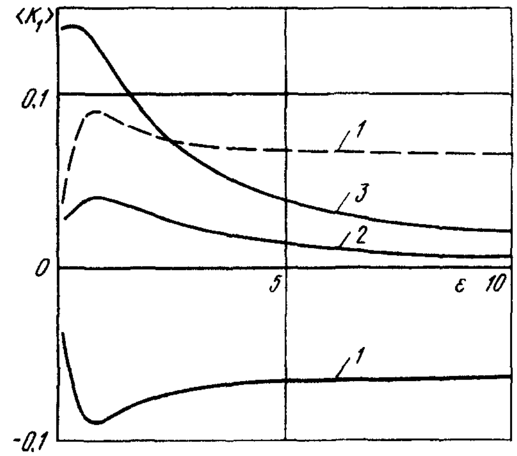

Fig. 2.

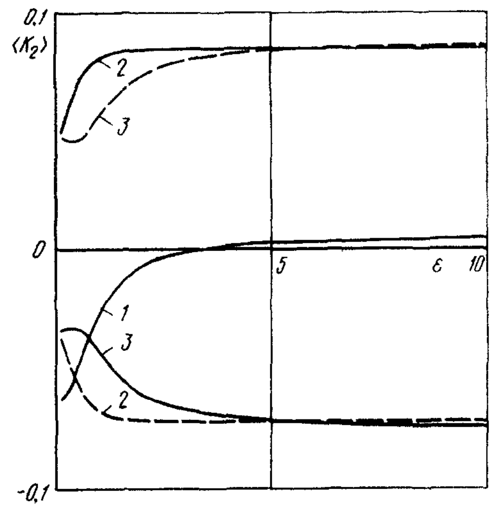

Fig. 3.

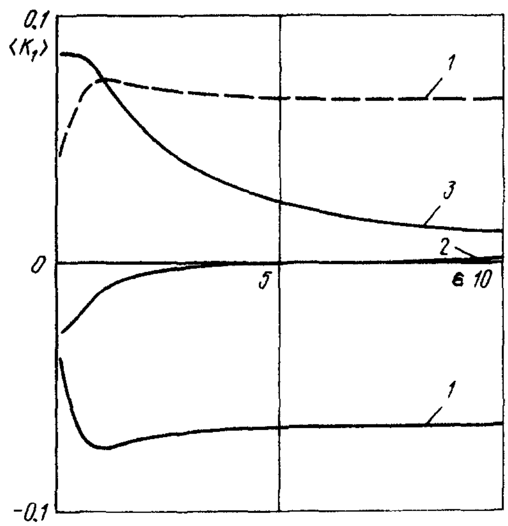

Fig. 4.

REFERENCES

1. BELOKOPYTOVA L. V., IVANENKO O. A. and FIL'SHTINSKII L. A., A concentrated force or charge on a piezoceramic plate with a crack. In *Elasticity and Plasticity*, pp. 12–19. Izd. Leningrad Gos. Univ., Leningrad, 1986.
2. BELOKOPYTOVA L. V., IVANENKO O. A. and FIL'SHTINSKII L. A., Load transmission from an elastic edge to a semi-infinite piezoceramic plate. *Izv. Akad. Nauk ArmSSR, Ser. Mekhanika* **34**, 5, 41–51, 1981.
3. PARTON V. Z. and KUDRYAVTSEV B. A., *Electromagnetoelasticity in Piezoelectric Media and Electric Conductors*. Nauka, Moscow, 1988.
4. GRINCHENKO V. T., ULITKO A.F. and SHUL'GA A. N., *Electroelasticity. Mechanics of Coupled Fields in Structural Elements*, Vol 5. Naukova Dumka, Kiev, 1989.
5. MORSE P. M. and FESHBACH H., *Methods of Theoretical Physics*, Vol 1. McGraw-Hill, New York, 1958.
6. FIL'SHTINSKII L. A., Boundary-value problems of the theory of elasticity for an anisotropic half-plane weakened by a hole or cut. *Izv. Akad. Nauk SSSR, MTT* 6, 72–79, 1980.
7. MUSKHELISHVILI N. I., *Singular Integral Equations: Boundary-value Problems of the Theory of Functions and Some Applications in Mathematical Physics*. Fizmatgiz, Moscow, 1962.
8. CHEREPANOV G. P., *Brittle Fracture Mechanics*. Nauka, Moscow, 1974.
9. LEKHNITSKII S. G., *Anisotropic Plates*. Gostekhizdat, Moscow, 1957.

Translated by T.J.Z.